Ergodic Theory - Week 13

Course Instructor: Florian K. Richter
Teaching assistant: Konstantinos Tsinas

1 Entropy

P1. Calculate the entropy of any periodic system.

> For any finite partition ξ and for any $n \geq k$ we have that
>
> $$H\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) = H\Big(\bigvee_{i=0}^{k-1} T^{-i}\xi\Big).$$
>
> Therefore, $h(T,\xi) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{k-1} T^{-i}\xi\Big) = 0$. We conclude that $h(T) = 0$.

P2. **(a)** Calculate the entropy of a rotation by any $\alpha \in \mathbb{T}$ on the torus $\mathbb{T}$.

> We consider two cases. First $\alpha \in \mathbb{Q}$, for which we have that there exists $k \in \mathbb{N}$ such that $T^k = T$ and thus by question 1 we have that the entropy of the system is 0. If $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ we consider the partition $\xi = \{[0, \frac{1}{2}), [\frac{1}{2}, 1)\}$. Irrational rotations are dense, and so countably many $R_\alpha^{-n}\xi$ generate any open interval, and since any Borel measurable set is generated by countably many open intervals, it follows that ξ is a generating partition. Therefore, by the Kolmogorov-Sinai Theorem, we have
>
> $$h(R_\alpha) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} R_\alpha^{-n}\xi\Big)$$
>
> Observe now that for any $n \in \mathbb{N}$, $\bigvee_{i=0}^{n-1} R_\alpha^{-n}\xi$ has $2n$ atoms, and then
>
> $$h(R_\alpha) \leq \lim_{n\to\infty} \frac{\log 2n}{n} = 0.$$

(b) Calculate the entropy of a rotation by any $\alpha \in \mathbb{T}^d$ on the d-dimensional torus $\mathbb{T}^d$, for any $d \in \mathbb{N}$.

> We show that the entropy of the product is the sum of the entropies of the base systems. Since a rotation in $\mathbb{T}^d$ is the product of d rotations in $\mathbb{T}$, we conclude that the entropy is 0 in this case as well.
>
> In order to prove our claim, consider two systems $(X, \mathcal{B}, \mu, T), (Y, \mathcal{A}, \nu, S)$ and let $\mathcal{S}$ be the algebra of rectangles (i.e. sets of the for $A_1 \times A_2$) in $X \times Y$, which generates the σ-algebra $\mathcal{A} \times \mathcal{B}$ by the definition of the product σ-algebra. Define
>
> $$h^*(T \times S) = \sup_\xi \{h(T, \xi) \colon \text{ all elements of the partition belong to } \mathcal{S}\}.$$

It is immediate that $h^*(T \times S) \leq h(T \times S)$, since we are taking the supremum over a smaller collection of partitions. We claim that $h^*(T \times S) = h(T \times S)$. Indeed, let $\xi = \{A_1, \ldots, A_k\}$ be any finite partition of the product σ-algebra, for every $1 \leq i \leq r$, consider sets $B_i \in \mathcal{S}$ such that $\mu(A_i \triangle B_i) < \varepsilon$. The sets B_i might not form a partition, so we modify them slightly by removing any common elements. Notice that if $i \neq j$, we have that $\mu(B_i \cap B_j) < 2\varepsilon$, since $B_i \cap B_j \subseteq (B_i \triangle A_i) \cup (B_j \triangle A_j)$. Let $C = \cup_{i \neq j} B_i \cap B_j$, so that $\mu(C) < r^2\varepsilon$. We set $D_i = B_i \setminus C$ for $1 \leq i \leq r-1$ and $D_r = X \setminus \cup_{i=0}^{r-1} D_i$. This induces a partition ξ' of X and if $i \leq r-1$, we have $D_i \triangle A_i \subseteq (B_i \triangle A_i) \cup N$, so that

$$\mu(A_i \triangle D_i) < (r^2+1)\varepsilon.$$

Additionally, $D_r \triangle A_r \subseteq \cup_{i=1}^{r-1}(D_i \triangle A_i)$, which implies that $\mu(A_r \triangle D_r) < (r-1)(r^2+1)\varepsilon$. We conclude that

$$\sum_{i=1}^{r-1} \mu(A_i \triangle D_i) < 2(r-1)(r^2+`1)\varepsilon.$$

Fix $\delta > 0$. By picking ε sufficiently small, we deduce (from Exercise 3 in Exercise sheet 12) that $H(\xi|\xi') < \delta$. We infer that $h(T \times S, \xi) \leq h(T \times S, \xi') + H(\xi|\xi') \leq h(T \times S, \xi') + \delta$. To see why the last inequality is true, we note that

$$\begin{aligned}
H\left(\bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi\right) \leq H\left(\bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi \vee \bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi'\right) = H\left(\bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi'\right) + \\
H\left(\bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi | \bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi'\right) \leq H\left(\bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi'\right) + \\
\sum_{i=0}^{n-1} H\left((T \times S)^{-i}\xi | \bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi'\right) \leq H\left(\bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi'\right) + \\
\sum_{i=0}^{n-1} H\left((T \times S)^{-i}\xi \big| (T \times S)^{-i}\xi'\right) = H\left(\bigvee_{i=0}^{n-1}(T \times S)^{-i}\xi'\right) + nH(\xi|\xi').
\end{aligned}$$

The desired inequality follows by dividing by n and taking $n \to +\infty$.

We have shown that for any finite partition of $\mathcal{B} \times \mathcal{A}$, we can find a partition consisting of rectangles and such that the entropies of the two partitions are very close together. We deduce that $h^*(T \times S) = h(T \times S)$

We prove that $h^*(T \times S) = h(T) + h(S)$. This follows from the fact that

$$h(T \times S, \xi_1 \times \xi_2) = H(T, \xi_1) + H(S, \xi_2), \tag{1}$$

where $\xi_1 \times \xi_2 = \{A \times B \colon A \in \xi_1, B \in \xi_2\}$. For any partitions $\eta_1 = \{C_1, \ldots\}, \eta_2 = \{D_1, \ldots\}$ of $\mathcal{B}, \mathcal{A}$ respectively, we have that

$$\begin{aligned}
H(\eta_1 \times \eta_2) = -\sum_{i,j} \mu \times \nu(C_i \times D_j) \log\left(\mu \times \nu(C_i \times D_j)\right) = \\
-\sum_{i,j} \mu(C_i)\nu(D_j)\left(\log \mu(C_i) + \log \nu(D_j)\right) = -\sum_i \mu(C_i) \log \mu(C_i) - \sum_j \nu(D_j) \log \nu(D_j) = \\
H(\eta_1) + H(\eta_2).
\end{aligned}$$

We conclude that

$$H\left(\bigvee_{i=0}^{n-1}(T\times S)^{-i}(\xi_1\times\xi_2)\right)=H\left(\bigvee_{i=0}^{n-1}T^{-i}\xi_1\times\bigvee_{i=0}^{n-1}S^{-i}\xi_2]\right)=H\left(\bigvee_{i=0}^{n-1}T^{-i}\xi_1\right)+H\left(\bigvee_{i=0}^{n-1}S^{-i}\xi_2\right)$$

for all $n\in\mathbb{N}$. Dividing by n and taking $n\to+\infty$, we conclude that (1) holds.

P3. Calculate the entropy of the map $T_d : x\to dx$ mod 1 on the torus, for any $d\geq 2$. Conclude that any two distinct systems of the form $(\mathbb{T},\mathcal{B}(T),\lambda,T_d)$, $d\in\mathbb{N}$ are not isomorphic.

We consider the partition $\xi=\{[\frac{a}{d},\frac{a+1}{d}:a=0,1,...,d-1\}$ and observe that for any $n\in\mathbb{N}$, $\bigvee_{i=0}^{n-1}T_d^{-i}\xi=\{[\frac{a}{d^n},\frac{a+1}{d^n}):a=0,1,...,d^n-1\}$. The rational numbers in $[0,1)$ with powers of d as denominators are dense in $[0,1)$, therefore, ξ generates the sigma algebra. Then, by Kolmogorov-Sinai's Theorem, we have

$$h(T_d)=\lim_{n\to\infty}\frac{1}{n}H\Big(\bigvee_{i=0}^{n-1}T_d^{-i}\xi\Big).$$

On the other hand, notice that $\bigvee_{i=0}^{n-1}T_d^{-i}\xi$ consists on d^n intervals of measure $1/d^n$. Therefore, by theorem from the lecture notes we have that

$$h(T_d)=\lim_{n\to\infty}\frac{1}{n}\log(d^n)=\log(d).$$

Now, for any $d_1\neq d_2$, we have that the systems induced by the transformations T_{d_1} and T_{d_2} have different entropy and, thus, they cannot be isomorphic.

To prove the last claim, it suffices to show that if we have a factor map $\pi:(X,\mathcal{B},\mu,T)\to(Y,\mathcal{A},\nu,S)$, then $h(T)\geq h(S)$. If ξ is any finite partition of $\mathcal{A}$, then $\pi^{-1}(\xi)=\{\pi^{-1}(A)\colon A\in\xi\}$ is a finite partition of $\mathcal{B}$ (up to sets of measure 0). In addition, we have

$$h(S,\xi)=-\sum_{A\in\xi}\nu(A)\log\nu(A)=-\sum_{A\in\xi}\mu\left(\pi^{-1}(A)\right)\log\mu\left(\pi^{-1}(A)\right)=h(T,\pi^{-}(\xi)),$$

where we used the fact that $\mu(\pi^{-1}(A))=\nu(A)$. Taking the supremum over all finite partitions of $\mathcal{A}$, we deduce that $h(S)\leq h(T)$.

P4. Calculate the entropy of the Bernoulli shift on $X=\{0,1,...,k\}^{\mathbb{N}}$, where the probability measure in $\{0,...,k\}$ is defined by the probability vector $(p_0,p_1,...,p_k)$.

Consider the partition $\xi=\{A_j\}_{j=0}^k$, where, for each $j=0,1,...,k$,

$$A_j=\{x:x_1=j\}.$$

Notice that for $n\in\mathbb{N}$, $T^{-(n-1)}\xi$ is the partition consisting of sets with the n-coordinate fixed. Hence, any cylinder is generated by $\bigvee_{i=0}^{n-1}T^{-i}\xi$ for n big enough. Thus, ξ is a generating partition. Then, by Kolmogorov-Sinai's Theorem, we have

$$h(T)=H(T,\xi).$$

Now, notice that $\bigvee_{i=1}^{n-1}T^{-i}\xi$ and ξ are independent given that they fix disjoint set of coordinates. Thus

$$h(T) = H(\xi \mid \bigvee_{i=1}^{n-1} T^{-i}\xi) = H(\xi) = -\sum_{j=0}^{k} p_j \log(p_j).$$